12 Analytic Geometry: The Conic Sections

> *"It is impossible not to feel stirred at the thought of the emotions of men at certain historic moments of adventure and discovery. Such moments are also granted to students in the abstract regions of thought, and high among them must be placed the morning when Descartes lay in bed and invented the method of coordinate geometry."*
>
> **—Alfred North Whitehead**

When NASA scientists want to launch a probe to study the planets of our solar system, how do they plot the path of the probe through space? For example, the *Voyager* probe (which discovered the presence of active volcanoes on the moon Io) crashed into the giant planet Jupiter after traveling almost three billion miles! How did NASA scientists know where Jupiter would be when the probe arrived?

Scientists can accurately predict the location of any planet in our solar system at any future time because they know that the planets follow elliptical orbits around the Sun. In fact, http://www.jpl.nasa.gov/

the Sun is one of the two foci of each ellipse. An ellipse can look a lot like a circle, or it can appear longer and flatter. Which planets in our solar system have orbits which are most nearly circular? In this chapter's project, you will learn how to answer that question.

Learn more about the planets and the probes that have been sent out to study them at http://www.jpl.nasa.gov/.

In 1637 the great French philosopher and scientist René Descartes developed an idea that the nineteenth-century British philosopher John Stuart Mill described as "the greatest single step ever made in the progress of the exact sciences." Descartes combined the techniques of algebra with those of geometry and created a new field of study called **analytic geometry**. Analytic geometry enables us to apply algebraic methods and equations to the solution of problems in geometry and, conversely, to obtain geometric representations of algebraic equations.

We will first develop two simple but powerful devices: a formula for the distance between two points and a formula for the coordinates of the midpoint of a line segment. With these tools, we will demonstrate the power of analytic geometry by proving a number of general theorems from plane geometry.

The power of the methods of analytic geometry is also very well demonstrated, as we shall see in this chapter, in a study of the conic sections. We will find in the course of that study that (a) a geometric definition can be converted into an algebraic equation, and (b) an algebraic equation can be classified by the type of graph it represents.

12.1 The Distance and Midpoint Formulas

There is a useful formula that gives the distance $\overline{PQ}$ between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. In Figure 1a, we have shown the x-coordinate of a point as the distance of the point from the y-axis and the y-coordinate as its distance from the x-axis. Thus, we have labeled the horizontal segments x_1 and x_2 and the vertical segments y_1 and y_2. In Figure 1b we use the lengths from Figure 1a to indicate that $\overline{PR} = x_2 - x_1$ and $\overline{QR} = y_2 - y_1$. Since triangle PRQ is a right triangle, we can apply the Pythagorean theorem.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

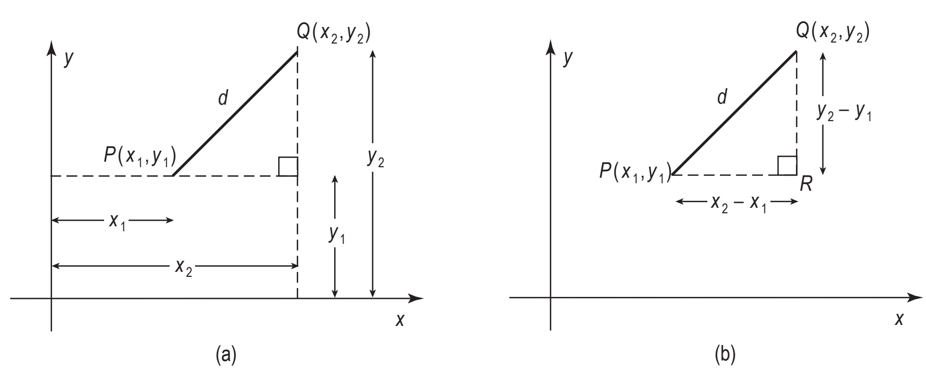


Figure 1 **Deriving the Distance Formula**

Although the points in Figure 1 are both in quadrant I, the same result will be obtained for any two points. Since distance cannot be negative, we have

The Distance Formula

The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The student should verify that the formula is true regardless of the quadrants in which P_1 and P_2 are located.

EXAMPLE 1 FINDING THE DISTANCE BETWEEN TWO POINTS

Find the distance between points $(3, -2)$ and $(-1, -5)$.

SOLUTION

We let $(x_1, y_1) = (3, -2)$ and $(x_2, y_2) = (-1, -5)$. Substituting these values in the distance formula, we have

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-1 - 3)^2 + [-5 - (-2)]^2} \\ &= \sqrt{(-4)^2 + (-3)^2} = \sqrt{25} = 5 \end{aligned}$$

If we had let $(x_1, y_1) = (-1, -5)$ and $(x_2, y_2) = (3, -2)$, we would have obtained the same result for d. Verify this.

✔ Progress Check 1

Find the distance between the points.

(a) $(-4, 3), (-2, 1)$ (b) $(-6, -7), (3, 0)$

Answers

(a) $2\sqrt{2}$ (b) $\sqrt{130}$

The Midpoint Formula

Another useful expression that is easily obtained is the one for the coordinates of the midpoint of a line segment. In Figure 2, we let $P(x, y)$ be the midpoint of the line segment whose endpoints are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let d denote the length of P_1P_2. Since P is the midpoint of P_1P_2, the length of P_1P is $d/2$. The lines PA and P_2B are parallel, so triangles P_1AP and P_1BP_2 are similar. Since corresponding sides of similar triangles are in proportion, we can write

$$\frac{\overline{P_1P_2}}{\overline{P_2B}} = \frac{\overline{P_1P}}{\overline{PA}}$$ where $\overline{P_1P_2}$ denotes the length of the segment P_1P_2

or

$$\frac{d}{y_2 - y_1} = \frac{\frac{d}{2}}{(y - y_1)}$$

This gives

$$\frac{d}{2}(y_2 - y_1) = d(y - y_1)$$

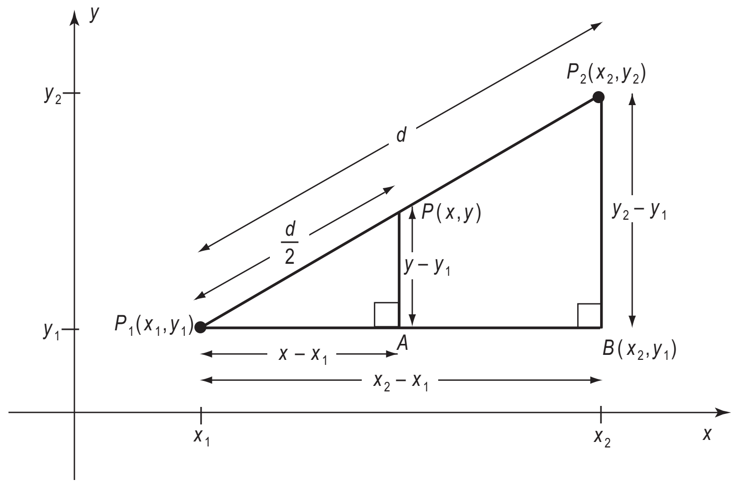


Figure 2 **Deriving the Midpoint Formula**

Dividing both sides by d, we have

$$\frac{1}{2}(y_2 - y_1) = (y - y_1)$$

$$\frac{1}{2}y_2 - \frac{1}{2}y_1 = y - y_1$$

$$y = \frac{y_1 + y_2}{2}$$

Similarly, from

$$\frac{\overline{P_1P_2}}{\overline{P_1B}} = \frac{\overline{P_1P}}{\overline{P_1A}}$$

we obtain

$$\frac{d}{x_2 - x_1} = \frac{\frac{d}{2}}{x - x_1}$$

so that

$$x = \frac{x_1 + x_2}{2}$$

We have the following general result:

The Midpoint Formula

If $P(x, y)$ is the midpoint of the line segment whose endpoints are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, then

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$

EXAMPLE 2 APPLYING THE MIDPOINT FORMULA

Find the coordinates of the midpoint of the line segment whose endpoints are $P_1(3, 4)$ and $P_2(-2, -6)$.

SOLUTION

If $P(x, y)$ is the midpoint, then

$$x = \frac{x_1 + x_2}{2} = \frac{3 - 2}{2} = \frac{1}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2} = \frac{4 - 6}{2} = -1$$

Thus, the midpoint is at $\left(\frac{1}{2}, -1\right)$.

✔ Progress Check 2

Find the coordinates of the midpoint of the line segment whose endpoints are given.

(a) $(0, -4), (-2, -2)$ (b) $(-10, 4), (7, -5)$

Answers

(a) $(-1, -3)$ (b) $\left(-\frac{3}{2}, -\frac{1}{2}\right)$

The formulas for distance, midpoint of a line segment, and slope of a line are adequate to allow us to demonstrate some of the beauty and power of analytic geometry. With these tools, we can prove many theorems from plane geometry by placing the figures on a rectangular coordinate system.

EXAMPLE 3 PROVING A THEOREM

Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and has length equal to one half the length of the third side.

SOLUTION

We place the triangle OAB in a convenient location, namely, with one vertex at the origin and one side on the positive x-axis (Figure 3). If Q and R are the midpoints of OB and AB, respectively, then by the midpoint formula the coordinates of Q are

$$\left(\frac{b}{2}, \frac{c}{2}\right)$$

and the coordinates of R are

$$\left(\frac{a+b}{2}, \frac{c}{2}\right)$$

We see that the line joining

$$Q\left(\frac{b}{2}, \frac{c}{2}\right) \quad \text{and} \quad R\left(\frac{a+b}{2}, \frac{c}{2}\right)$$

has slope 0 since the difference of the y-coordinates is

$$\frac{c}{2} - \frac{c}{2} = 0$$

Side OA also has slope 0, which proves that QR is parallel to OA.

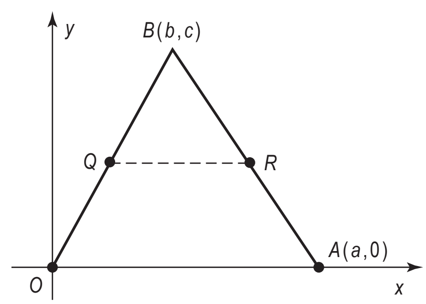


Figure 3 **Diagram for Example 3**

Applying the distance formula to QR, we have

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{\left(\frac{a+b}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2} \\ &= \sqrt{\left(\frac{a}{2}\right)^2} = \frac{a}{2} \end{aligned}$$

Since OA has length a, we have shown that $\overline{QR}$ is one half of $\overline{OA}$.

✔ Progress Check 3

Prove that the midpoint of the hypotenuse of a right triangle is equidistant from all three vertices.

Answer

Hint: Place the triangle so that two legs coincide with the positive x- and y-axes. Find the coordinates of the midpoint of the hypotenuse by the midpoint formula. Finally, compute the distance from the midpoint to each vertex by the distance formula.

Exercise Set 12.1

In Exercises 1–12 find the distance between the given points.

1. $(5, 4), (2, 1)$
2. $(-4, 5), (-2, 3)$
3. $(-1, -5), (-5, -1)$
4. $(2, -4), (3, -1)$
5. $(3, 1), (-4, 5)$
6. $(2, 4), (-3, 3)$
7. $(-2, 4), (-4, -2)$
8. $(-3, 0), (2, -4)$
9. $\left(-\frac{1}{2}, 3\right), \left(-1, -\frac{3}{4}\right)$
10. $(3, 0), (0, 4)$
11. $(2, -4), (0, -1)$
12. $\left(\frac{2}{3}, \frac{3}{2}\right), (-2, -4)$

In Exercises 13–24 find the midpoint of the line segment whose endpoints are the given pair of points.

13. $(2, 6), (3, 4)$
14. $(1, 1), (-2, 5)$
15. $(2, 0), (0, 5)$
16. $(-3, 0), (-5, 2)$
17. $(-2, 1), (-5, -3)$
18. $(2, 3), (-1, 3)$
19. $(0, -4), (0, 3)$
20. $(1, -3), (3, 2)$
21. $(-1, 3), (-1, 6)$
22. $(3, 2), (0, 0)$
23. $(1, -1), (-1, 1)$
24. $(2, 4), (2, -4)$

*25. Show that the medians from the equal angles of an isosceles triangle are of equal length. (*Hint*: Place the triangle so that its vertices are at the points $A(-a, 0)$, $B(a, 0)$, and $C(0, b)$.)

*26. Show that the midpoints of the sides of a rectangle are the vertices of a rhombus (a quadrilateral with four equal sides). (*Hint*: Place the rectangle so that its vertices are at the points $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b).)

*27. Show that a triangle with two equal medians is isosceles.

*28. Show that the sum of the squares of the lengths of the medians of a triangle equals three fourths the sum of the squares of the lengths of the sides. (*Hint*: Place the triangle so that its vertices are the points $(-a, 0)$, $(b, 0)$, and $(0, c)$.)

*29. Show that the diagonals of a rectangle are equal in length. (*Hint*: Place the rectangle so that its vertices are the points $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b).)

*30. Find the length of the longest side of the triangle whose vertices are $A(3, -4)$, $B(-2, -6)$, and $C(-1, 2)$.

12.2 Symmetry

If we fold the graph in Figure 4a along the x-axis, the portion of the graph lying above the x-axis will coincide with the portion lying below. Similarly, if we fold the graph in Figure 4b along the y-axis, the portion to the left of the y-axis will coincide with the portion to the right. These properties illustrate the notion of symmetry, which we now define more carefully.

A curve in the xy-plane is **symmetric with respect to the**

(a) **x-axis** if for every point (x_1, y_1) on the curve, the point $(x_1, -y_1)$ is also on the curve;

(b) **y-axis** if for every point (x_1, y_1) on the curve, the point $(-x_1, y_1)$ is also on the curve;

(c) **origin** if for every point (x_1, y_1) on the curve, the point $(-x_1, -y_1)$ is also on the curve.

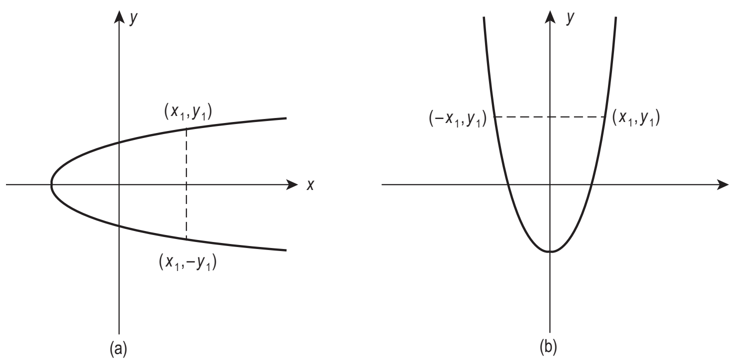


Figure 4 **Symmetry with Respect to the Coordinate Axes**

Thus, a curve is symmetric with respect to the x-axis if the portion of the curve lying below the x-axis is the mirror image in the x-axis of the portion above the x-axis. Similarly, a curve is symmetric with respect to the y-axis if the portion of the curve lying to the left of the y-axis is the mirror image in the y-axis of the portion to the right of the y-axis. Thus, the curve in Figure 4a is symmetric with respect to the x-axis, and the curve in Figure 4b is symmetric with respect to the y-axis. The curve in Figure 5 is symmetric with respect to the origin.

The symmetries of a curve can be discovered by looking at the curve. However, it is sometimes helpful to discover the symmetries of the graph by examining the equation and to use these symmetries as aids in sketching the graph. Thus, we have the following tests for symmetry:

Tests for Symmetry

The graph of an equation is **symmetric with respect to the**

(a) **x-axis** if replacing y with $-y$ results in an equivalent equation;
(b) **y-axis** if replacing x with $-x$ results in an equivalent equation;
(c) **origin** if replacing x with $-x$ and y with $-y$ results in an equivalent equation.

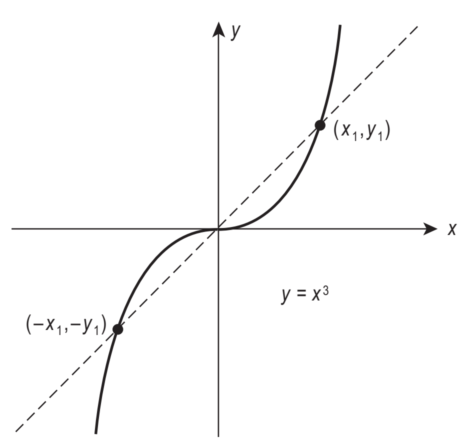


Figure 5 **Symmetry with Respect to the Origin**

EXAMPLE 1 DETERMINING SYMMETRY

Without sketching the graph, determine symmetry with respect to the x- and y-axes.

(a) $x^2 + 4y^2 - y = 1$ (b) $xy = 5$

SOLUTIONS

(a) Replacing x by $-x$, we have

$$\begin{aligned} (-x)^2 + 4y^2 - y &= 1 \\ x^2 + 4y^2 - y &= 1 \end{aligned}$$

Since this is an equivalent equation, the curve is symmetric with respect to the y-axis. Now, replacing y by $-y$, we have

$$\begin{aligned} x^2 + 4(-y)^2 - (-y) &= 1 \\ x^2 + 4y^2 + y &= 1 \end{aligned}$$

which is *not* an equivalent equation. Thus, the curve is not symmetric with respect to the x-axis.

(b) Replacing x by $-x$, we have $-xy = 5$, which is not an equivalent equation. Replacing y by $-y$, we have $-xy = 5$, which is not an equivalent equation. Thus, the curve is not symmetric with respect to either axis.

✔ Progress Check 1

Without graphing, determine symmetry with respect to the coordinate axes.

(a) $x^2 - y^2 = 1$ (b) $x + y = 10$ (c) $y = \dfrac{1}{x^2 + 1}$

Answers

(a) symmetric with respect to both x- and y-axes
(b) not symmetric with respect to either axis
(c) symmetric with respect to the y-axis

EXAMPLE 2 DETERMINING SYMMETRY

Determine symmetry with respect to the origin.

(a) $y = x^3 - 1$ (b) $y^2 = \dfrac{x^2 + 1}{x^2 - 1}$

SOLUTIONS

(a) Replacing x by $-x$ and y by $-y$, we have

$$\begin{aligned} -y &= (-x)^3 - 1 \\ -y &= -x^3 - 1 \\ y &= x^3 + 1 \end{aligned}$$

Since the equation is not an equivalent equation, the curve is not symmetric with respect to the origin.

(b) Replacing x by $-x$ and y by $-y$, we have

$$\begin{aligned} (-y)^2 &= \frac{(-x)^2 + 1}{(-x)^2 - 1} \\ y^2 &= \frac{x^2 + 1}{x^2 - 1} \end{aligned}$$

The equation is an equivalent equation, so we conclude that the curve is symmetric with respect to the origin.

✔ Progress Check 2

Determine symmetry with respect to the origin.

(a) $x^2 + y^2 = 1$ (b) $y^2 = x - 1$ (c) $y = x + \dfrac{1}{x}$

Answers

(a) symmetric with respect to the origin
(b) not symmetric with respect to the origin
(c) symmetric with respect to the origin

Note that in Example 2b and Progress Check 2a, the given curves are symmetric with respect to *both* the x- and y-axes, as well as the origin. In fact, we have the following general rule:

> A curve that is symmetric with respect to both coordinate axes is also symmetric with respect to the origin. The converse, however, is not true.

The curve $y = x^3$ in Figure 5 is an example of one that is symmetric with respect to the origin, but not with respect to the coordinate axes.

Exercise Set 12.2

In Exercises 1–34 determine, without graphing, whether the given curve is symmetric with respect to the x-axis, the y-axis, or the origin, or is not symmetric with respect to any of these.

1. $y = x + 3$
2. $x - 2y = 5$
3. $2x - 5y = 0$
4. $3x + 2y = 5$
5. $y = 4x^2$
6. $x = 2y^2$
7. $y^2 = x - 4$
8. $x^2 - y = 2$
9. $y = 4 - 9x^2$
10. $y = x(x - 4)$
11. $y = 1 + x^3$
12. $y^2 = 1 + x^3$
13. $y = (x - 2)^2$
14. $y^2 = (x - 2)^2$
15. $y^2 = x^2 - 9$
16. $y^2x + 2x = 4$
17. $y + yx^2 = x$
18. $y^2x + 2x^2 = 4x^2y$
19. $y^3 = x^2 - 9$
20. $y^3 = x^3 + 9$
21. $y = \dfrac{1}{x^2}$
22. $y = \dfrac{x^2 + 4}{x^2 - 4}$
23. $y = \dfrac{1}{x^2 + 1}$
24. $y^2 = \dfrac{x^2 + 1}{x^2 - 1}$
25. $4y^2 - x^2 = 1$
26. $4x^2 + 9y^2 = 36$
27. $9y^2 - 4x^2 = 36$
28. $4 + x^2y = x^2$
29. $y^2 = \dfrac{1}{x^2}$
30. $y = \dfrac{1}{x^3}$
31. $y = x + \dfrac{1}{x^2}$
32. $y^2 = \dfrac{x}{x - 1}$
33. $xy = 4$
34. $x^2y = 4$

*35. Show that the graph of an even function is symmetric with respect to the y-axis. (See Exercise 61 in Section 6.2.)

*36. Show that the graph of an odd function is symmetric with respect to the origin. (See Exercise 61 in Section 6.2.)

12.3 The Circle

The conic sections provide us with an outstanding opportunity to demonstrate the double-edged power of analytic geometry. We will see that a geometric figure defined as a set of points can often be described analytically by an algebraic equation; conversely, we can start with an algebraic equation and use graphing procedures to study the properties of the curve.

First, let's see how the term "conic section" originates. If we pass a plane through a cone at various angles, the intersections are called conic sections. Figure 6 shows four conic sections: a circle, a parabola, an ellipse, and a hyperbola.

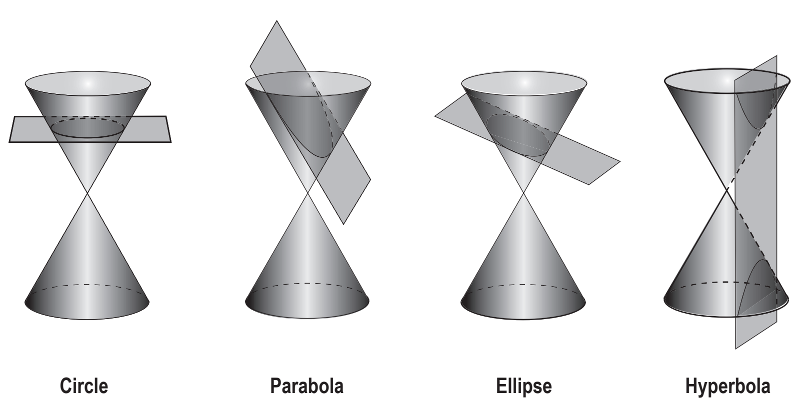


Figure 6 The Conic Sections

Let's begin with the geometric definition of a circle.

A **circle** is the set of all points in a plane that are at a given distance from a fixed point. The fixed point is called the **center** of the circle, and the given distance is called the **radius**.

Using the methods of analytic geometry, we place the center at a point (h, k), as in Figure 7. If $P(x, y)$ is a point on the circle, then the distance from P to the center (h, k) must be equal to the radius r. By the distance formula

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

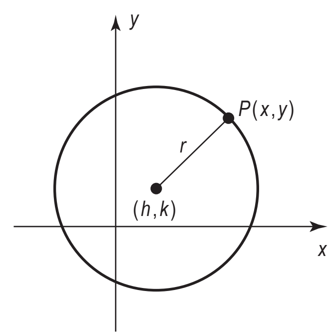


Figure 7 **Deriving the Equation of the Circle**

Squaring both sides provides us with an important form of the equation of the circle.

Standard Form of the Equation of a Circle

$$(x-h)^2 + (y-k)^2 = r^2$$

is the standard form of the equation of the circle with center at (h, k) and radius r.

EXAMPLE 1 FINDING THE EQUATION OF A CIRCLE

Write the standard form of the equation of the circle with center at $(2, -5)$ and radius 3.

SOLUTION

Substituting $h = 2$, $k = -5$, and $r = 3$ in the equation

$$(x-h)^2 + (y-k)^2 = r^2$$

yields

$$(x-2)^2 + (y+5)^2 = 9$$

✔ Progress Check 1

Write the standard form of the equation of the circle with center at $(-4, -6)$ and radius 5.

Answer

$(x + 4)^2 + (y + 6)^2 = 25$

EXAMPLE 2 USING THE STANDARD FORM FOR A CIRCLE

Find the coordinates of the center and the radius of the circle whose equation is

$$(x + 1)^2 + (y - 3)^2 = 4$$

SOLUTION

Since the standard form is

$$(x - h)^2 + (y - k)^2 = r^2$$

we must have

$$x - h = x + 1, \qquad y - k = y - 3, \qquad r^2 = 4$$

Solving, we find that

$$h = -1, \qquad k = 3, \qquad r = 2$$

Thus, the center is at $(-1, 3)$ and the radius is 2.

✔ Progress Check 2

Find the coordinates of the center and the radius of the circle whose equation is $(x - \frac{1}{2})^2 + (y + 5)^2 = 15$.

Answer

center: $(\frac{1}{2}, -5)$; radius: $\sqrt{15}$

General Form

It is also possible to begin with the equation of a circle in the **general form**

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \quad A \neq 0$$

and to rewrite the equation in standard form. The process involves completing the square in each variable.

EXAMPLE 3 WRITING AN EQUATION IN STANDARD FORM

Write the equation of the circle $2x^2 + 2y^2 - 12x + 16y - 31 = 0$ in standard form.

SOLUTION

Grouping the terms in x and y and factoring produces

$$2(x^2 - 6x) + 2(y^2 + 8y) = 31$$

Completing the square in both x and y, we have

$$2(x^2 - 6x + 9) + 2(y^2 + 8y + 16) = 31 + 18 + 32$$
$$2(x - 3)^2 + 2(y + 4)^2 = 81$$

Note that the quantities 18 and 32 were added to the right-hand side because each factor is multiplied by 2. The last equation can be written as

$$(x - 3)^2 + (y + 4)^2 = \frac{81}{2}$$

This is the standard form of the equation of the circle with center at $(3, -4)$ and radius $\dfrac{9\sqrt{2}}{2}$.

✔ Progress Check 3

Write the equation of the circle $4x^2 + 4y^2 - 8x + 4y = 103$ in standard form, and determine the center and radius.

Answer

$(x - 1)^2 + (y + \frac{1}{2})^2 = 27$; center: $(1, -\frac{1}{2})$; radius: $\sqrt{27}$

EXAMPLE 4 WRITING AN EQUATION IN STANDARD FORM

Write the equation $3x^2 + 3y^2 - 6x + 15 = 0$ in standard form.

SOLUTION

Grouping and factoring, we have

$$3(x^2 - 2x) + 3y^2 = -15$$

We then complete the square in x and y:

$$3(x^2 - 2x + 1) + 3y^2 = -15 + 3$$
$$3(x - 1)^2 + 3y^2 = -12$$
$$(x - 1)^2 + y^2 = -4$$

Since $r^2 = -4$ is an impossible situation, the graph of the equation is not a circle. Note that the left-hand side of the equation in standard form is a sum of squares and is therefore nonnegative, while the right-hand side is negative. Thus, there are no real values of x and y that satisfy the equation. This is an example of an equation that does not have a graph.

✔ Progress Check 4

Write the equation $x^2 + y^2 - 12y + 36 = 0$ in standard form, and analyze its graph.

Answer

The standard form is $x^2 + (y - 6)^2 = 0$. The equation is that of a "circle" with center at (0, 6) and radius 0. The "circle" is actually the point (0, 6).

EXERCISE SET 12.3

In Exercises 1–8 write an equation of the circle with center at (h, k) and radius r.

1. $(h, k) = (2, 3),\ r = 2$
2. $(h, k) = (-3, 0),\ r = 3$
3. $(h, k) = (-2, -3),\ r = \sqrt{5}$
4. $(h, k) = (2, -4),\ r = 4$
5. $(h, k) = (0, 0),\ r = 3$
6. $(h, k) = (0, -3),\ r = 2$
7. $(h, k) = (-1, 4),\ r = 2\sqrt{2}$
8. $(h, k) = (2, 2),\ r = 2$

In Exercises 9–16 find the coordinates of the center and radius of the circle with the given equation.

9. $(x - 2)^2 + (y - 3)^2 = 16$
10. $(x + 2)^2 + y^2 = 9$
11. $(x - 2)^2 + (y + 2)^2 = 4$
12. $\left(x + \frac{1}{2}\right)^2 + (y - 2)^2 = 8$
13. $(x + 4)^2 + \left(y + \frac{3}{2}\right)^2 = 18$
14. $x^2 + (y - 2)^2 = 4$
15. $\left(x - \frac{1}{3}\right)^2 + y^2 = -\frac{1}{9}$
16. $x^2 + \left(y - \frac{1}{2}\right)^2 = 3$

In Exercises 17–24 write the equation of the given circle in standard form and determine the radius and the coordinates of the center, if possible.

17. $x^2 + y^2 + 4x - 8y + 4 = 0$
18. $x^2 + y^2 - 2x + 6y - 15 = 0$
19. $2x^2 + 2y^2 - 6x - 10y + 6 = 0$
20. $2x^2 + 2y^2 + 8x - 12y - 8 = 0$
21. $2x^2 + 2y^2 - 4x - 5 = 0$
22. $4x^2 + 4y^2 - 2y + 7 = 0$
23. $3x^2 + 3y^2 - 12x + 18y + 15 = 0$
24. $4x^2 + 4y^2 + 4x + 4y - 4 = 0$

In Exercises 25–32 write the equation in standard form, and determine if the graph of the equation is a circle, a point, or neither.

25. $x^2 + y^2 - 6x + 8y + 7 = 0$
26. $x^2 + y^2 + 4x + 6y + 5 = 0$
27. $x^2 + y^2 + 3x - 5y + 7 = 0$
28. $x^2 + y^2 - 4x - 6y - 13 = 0$
29. $2x^2 + 2y^2 - 12x - 4 = 0$
30. $2x^2 + 2y^2 + 4x - 4y + 25 = 0$
31. $2x^2 + 2y^2 - 6x - 4y - 2 = 0$
32. $2x^2 + 2y^2 - 10y + 6 = 0$

*33. Find the area of the circle whose equation is
$$x^2 + y^2 - 2x + 4y - 4 = 0$$

*34. Find the circumference of the circle whose equation is
$$x^2 + y^2 - 6x + 8 = 0$$

*35. Show that the circles whose equations are
$$x^2 + y^2 - 4x + 9y - 3 = 0$$
and
$$3x^2 + 3y^2 - 12x + 27y - 27 = 0$$
are concentric (have the same centers).

*36. Find an equation of the circle that has its center at (3, −1) and that passes through the point (−2, 2).

*37. Find an equation of the circle that has its center at (−5, 2) and that passes through the point (−3, 4).

*38. The two points (−2, 4) and (4, 2) are the endpoints of a diameter of a circle. Write the equation of the circle in standard form.

*39. The two points (3, 5) and (7, −3) are the endpoints of a diameter of a circle. Write the equation of the circle in standard form.

12.4 The Parabola

We begin our study of the parabola with the geometric definition.

A **parabola** is the set of all points that are equidistant from a given point and a given line.

The given point is called the **focus** and the given line is called the **directrix** of the parabola. In Figure 8, all points P on the parabola are equidistant from the focus F and the directrix L; that is, $\overline{PF} = \overline{PQ}$. The line through the focus that is perpendicular to the directrix is called the **axis of the parabola** (or simply the **axis**), and the parabola is seen to be symmetric with respect to the axis. The point V (Figure 8), where the parabola intersects its axis, is called the **vertex** of the parabola. The vertex, then, is the point from which the parabola opens. Note that the vertex is the point on the parabola that is closest to the directrix.

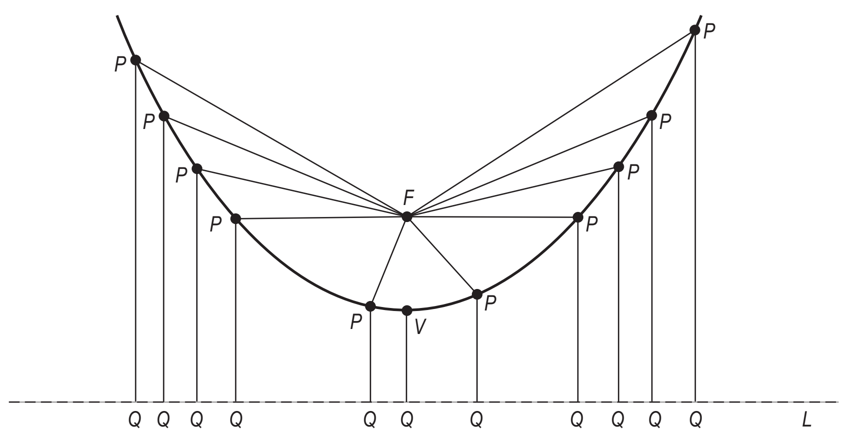


Figure 8 **Deriving the Equation of the Parabola**

We can apply the methods of analytic geometry to find an equation of the parabola. We choose the y-axis as the axis of the parabola and the origin as the vertex (Figure 9). Since the vertex is on the parabola, it is equidistant from the focus and the directrix. Thus, if the coordinates of the focus F are $(0, p)$, then the equation of the directrix is $y = -p$. We then let $P(x, y)$ be any point on the parabola, and we equate the distance from P to the focus F and the distance from P to the directrix L. Using the distance formula, we have

$$\overline{PF} = \overline{PQ}$$
$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{(x-x)^2 + (y+p)^2}$$

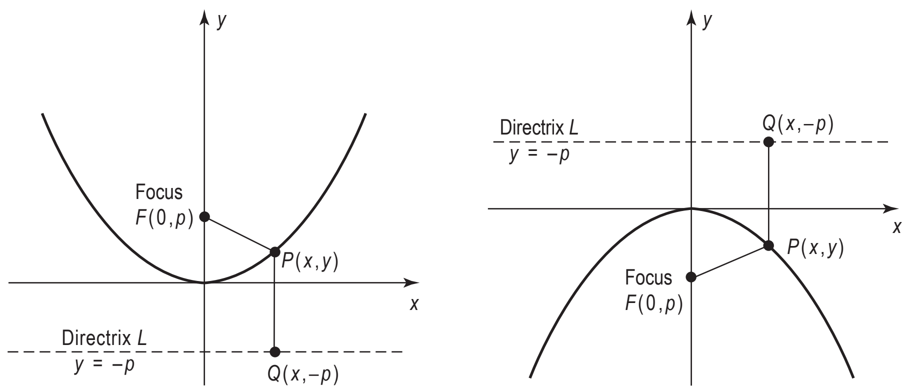
Squaring both sides, and expanding, we have

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$
$$x^2 = 4py$$

We have obtained an important form of the equation of a parabola.

> $$x^2 = 4py$$
> is the equation of a parabola whose vertex is at the origin, whose focus is at $(0, p)$, and whose axis is vertical.

Conversely, it can be shown that the graph of the equation $x^2 = 4py$ is a parabola. Note that substituting $-x$ for x leaves the equation unchanged, verifying symmetry with respect to the y-axis. If $p > 0$, the parabola opens upward, as shown in Figure 9a; if $p < 0$, the parabola opens downward, as shown in Figure 9b.



(a) $p > 0$ (b) $p < 0$

Figure 9 Parabolas with Vertex at the Origin and a Vertical Axis

EXAMPLE 1 SKETCHING THE GRAPH OF A PARABOLA

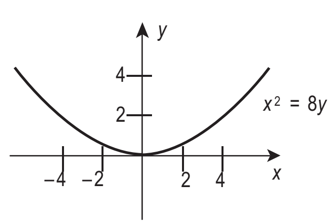
Sketch the graph of each equation.

(a) $x^2 = 8y$ (b) $x^2 = -2y$

SOLUTIONS

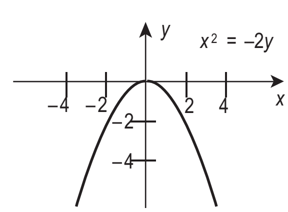
We form tables of values giving points on the graphs and draw smooth curves. See Figure 10.

y	x
0	0
1	±2.83
2	±4
3	±4.9



(a)

y	x
0	0
−1	±1.41
−3	±2.45
3	±3.16



(b)

Figure 10 Diagram for Example 1

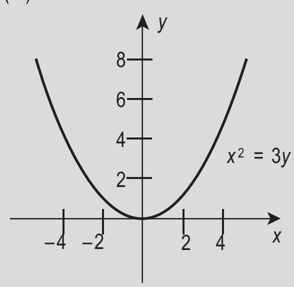
✔ Progress Check 1

Sketch the graph of each equation.

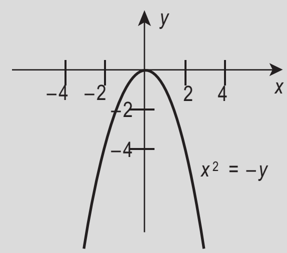
(a) $x^2 = 3y$ (b) $x^2 = -y$

Answers

(a)



(b)



If we place the parabola as shown in Figure 11, we can proceed as before to obtain the following result:

$$y^2 = 4px$$

is the equation of a parabola whose vertex is at the origin, whose focus is at $(p, 0)$, and whose axis is horizontal.

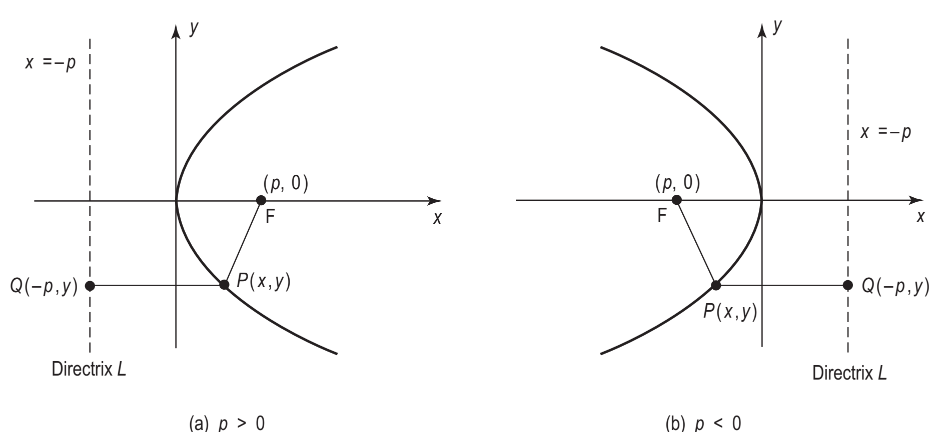


Figure 11 **Parabolas with Vertex at the Origin and a Horizontal Axis**

Note that substituting $-y$ for y leaves this equation unchanged, verifying symmetry with respect to the x-axis. If $p > 0$, the parabola opens to the right, as shown in Figure 11a; if $p < 0$, the parabola opens to the left, as shown in Figure 11b.

EXAMPLE 2 SKETCHING THE GRAPH OF A PARABOLA

Sketch the graph of each equation.

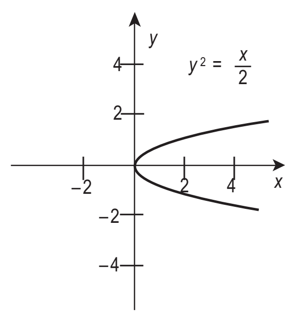
(a) $y^2 = \dfrac{x}{2}$ (b) $y^2 = -2x$

SOLUTIONS

We form tables of values giving points on the graphs and draw smooth curves. See Figure 12.

(a)

x	y
0	0
1	±0.71
3	±1.22
5	±1.58



(b)

x	y
0	0
−1	±1.41
−3	±2.45
−5	±3.16

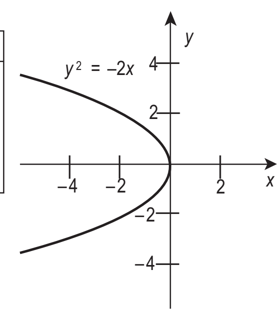


Figure 12 **Diagram for Example 2**

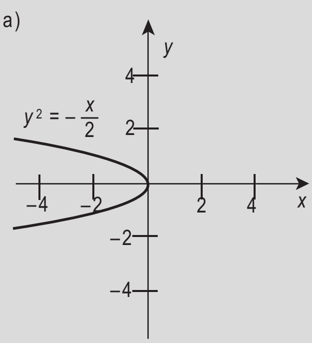
✔ Progress Check 2

Sketch the graph of each equation.

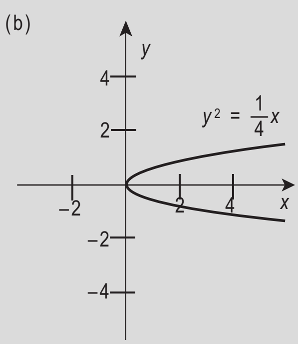
(a) $y^2 = -\dfrac{x}{2}$ (b) $y^2 = \dfrac{1}{4}x$

Answers

(a)



(b)



EXAMPLE 3 FINDING THE EQUATION OF A PARABOLA

Find the equation of the parabola that has the x-axis as its axis, has vertex at (0, 0), and passes through the point $(-2, 3)$.

SOLUTION

Since the axis of the parabola is the x-axis, the equation of the parabola is $y^2 = 4px$. The parabola passes through the point $(-2, 3)$, so the coordinates of this point must satisfy the equation of the parabola. Thus,

$$y^2 = 4px$$
$$(3)^2 = 4p(-2)$$
$$4p = -\frac{9}{2}$$

and the equation of the parabolas is

$$y^2 = 4px = -\frac{9}{2}x$$

✔ Progress Check 3

Find the equation of the parabola that has the x-axis as its axis, has vertex at (0, 0), and passes through the point $(-2, 1)$.

Answer

$y^2 = -\frac{1}{2}x$

Vertex at (*h, k*)

It is also possible to determine an equation of the parabola when the vertex is at some arbitrary point (h, k). The form of the equation depends on whether the axis of the parabola is parallel to the x-axis or to the y-axis. The situations are summarized in Table 1. Note that if the point (h, k) is the origin, then $h = k = 0$, and we arrive at the equations we derived previously, $x^2 = 4py$ and $y^2 = 4px$.

TABLE 1 Standard Forms of the Equations of the Parabola

Equation	Vertex	Axis
$(x - h)^2 = 4p(y - k)$	(h, k)	$x = h$
$(y - k)^2 = 4p(x - h)$	(h, k)	$y = k$

EXAMPLE 4 GRAPHING FROM STANDARD FORM

Sketch the graph of the equation $(y - 3)^2 = -2(x + 2)$.

SOLUTION

The equation is the standard form of a parabola, with vertex at $(-2, 3)$ and axis of symmetry $y = 3$. See Figure 13.

y	x
5	-4
3	-2
1	-4
0	$-\frac{13}{2}$

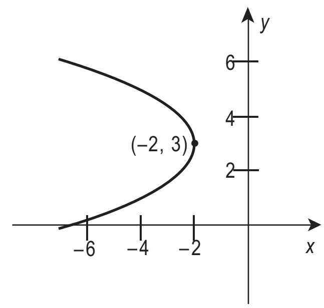
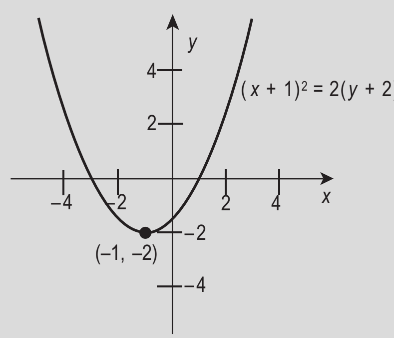


Figure 13 **Diagram for Example 4**

✔ Progress Check 4

Sketch the graph of the equation $(x + 1)^2 = 2(y + 2)$. Locate the vertex and the axis of symmetry.

Answer



vertex: $(-1, -2)$; axis of symmetry: $x = -1$

From the graphs in Figures 10 and 13 and the answers to Progress Checks 1 and 4, we can make the following observations:

The graph of a parabola whose equation is

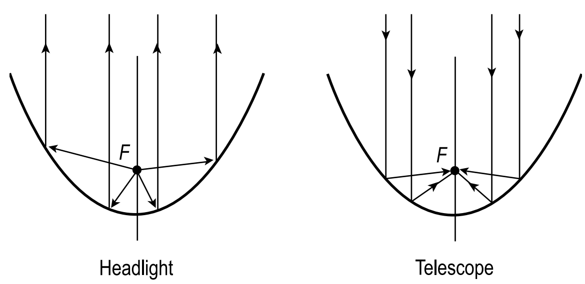
$$(x - h)^2 = 4p(y - k)$$

opens upward if $p > 0$ and downward if $p < 0$, and the axis of symmetry is $x = h$. See Figures 14a and 14b.

Devices with a Parabolic Shape

The properties of the parabola are used in the design of some important devices. For example, by rotating a parabola about its axis, we obtain a **parabolic reflector**, a shape used in the headlight of an automobile. In the accompanying figure, the light source (the bulb) is placed at the focus of the parabolic reflector. The headlight is coated with a reflecting material, and the rays of light bounce back in lines that are parallel to the axis of the parabola. This permits a headlight to disperse light in front of the auto where it is needed.

A reflecting telescope reverses the use of these same properties. Here, the rays of light from a distant star, which are nearly parallel to the axis of the parabola, are reflected by the mirror to the focus (see accompanying figure). The eyepiece is placed at the focus, where the rays of light are gathered.



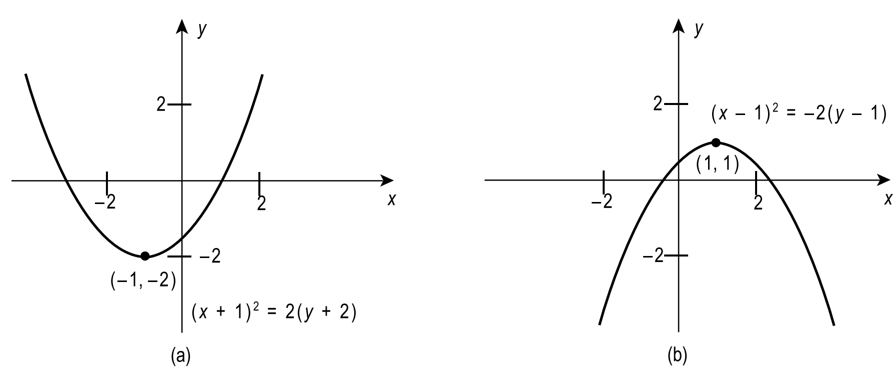


Figure 14 **Parabola with Center at (h, k) and Vertical Axis**

The graph of a parabola whose equation is

$$(y - k)^2 = 4p(x - h)$$

opens to the right if $p > 0$ and to the left if $p < 0$, and the axis of symmetry is $y = k$. See Figures 15a and 15b.

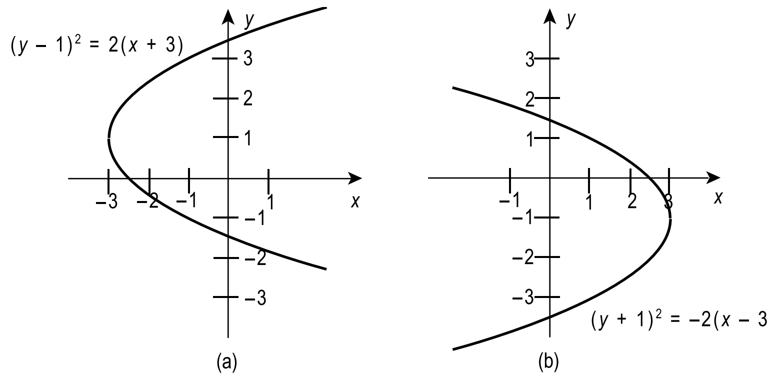


Figure 15 **Parabolas with Center at (h, k) and Horizontal Axis**

EXAMPLE 5 WORKING WITH THE STANDARD FORM

Determine the vertex, axis, and direction of opening of the graph of the parabola

$$\left(x - \frac{1}{2}\right)^2 = -\frac{1}{2}(y + 4)$$

SOLUTION

Comparison of the equation with the standard form

$$(x - h)^2 = 4p(y - k)$$

yields

$$h = \frac{1}{2}, k = -4, p = -\frac{1}{8}$$

The axis of the parabola is always found by setting the square term equal to 0.

$$\left(x - \frac{1}{2}\right)^2 = 0$$

$$x = \frac{1}{2}$$

Thus, the vertex is $(h, k) = \left(\frac{1}{2}, -4\right)$, the axis is $x = \frac{1}{2}$, and the parabola opens downward since $p < 0$.

✔ Progress Check 5

Determine the vertex, axis, and direction of opening of the graph of the parabola

$$(y + 1)^2 = 4\left(x - \frac{1}{3}\right)$$

Answer

vertex: $(\frac{1}{3}, -1)$; axis: $y = -1$; opens to the right

Any second-degree equation in x and y that has a square term in one variable but only first-degree terms in the other, represents a parabola. We can put such an equation in standard form by completing the square.

EXAMPLE 6 WRITING AN EQUATION IN STANDARD FORM

Determine the vertex, axis, and direction of opening of the parabola

$$2y^2 - 12y + x + 19 = 0$$

SOLUTION

First, we complete the square in y.

$$\begin{aligned} 2y^2 - 12y + x + 19 &= 0 \\ 2(y^2 - 6y) &= -x - 19 \\ 2(y^2 - 6y + 9) &= -x - 19 + 18 \\ 2(y - 3)^2 &= -x - 1 = -(x + 1) \\ (y - 3)^2 &= -\frac{1}{2}(x + 1) \end{aligned}$$

With the equation in standard form, we see that $(h, k) = (-1, 3)$ is the vertex, $y = 3$ is the axis, and the curve opens to the left.

✔ Progress Check 6

Write the equation of the parabola $x^2 + 4x + y + 9 = 0$ in standard form. Determine the vertex, axis, and direction of opening.

Answer

$(x + 2)^2 = -(y + 5)$; vertex: $(-2, -5)$; axis: $x = -2$; opens downward

Exercise Set 12.4

In Exercises 1–16 sketch the graph of the given equation.

1. $x^2 = 4y$
2. $x^2 = -4y$
3. $y^2 = 2x$
4. $y^2 = -\frac{3}{2}x$
5. $x^2 = y$
6. $y^2 = x$
7. $x^2 + 5y = 0$
8. $2y^2 - 3x = 0$
9. $(x - 2)^2 = 2(y + 1)$
10. $(x + 4)^2 = 3(y - 2)$
11. $(y - 1)^2 = 3(x - 2)$
12. $(y - 2)^2 = -2(x + 1)$
13. $(x + 4)^2 = -\frac{1}{2}(y + 2)$
14. $(y - 1)^2 = -3(x - 2)$
15. $y^2 = -2(x + 1)$
16. $x^2 = \frac{1}{2}(y - 3)$

In Exercises 17–30 determine the vertex, axis, and direction of opening of the given parabola.

17. $x^2 - 2x - 3y + 7 = 0$
18. $x^2 + 4x + 2y - 2 = 0$
19. $y^2 - 8y + 2x + 12 = 0$
20. $y^2 + 6y - 3x + 12 = 0$
21. $x^2 - x + 3y + 1 = 0$
22. $y^2 + 2y - 4x - 3 = 0$
23. $y^2 - 10y - 3x + 24 = 0$
24. $x^2 + 2x - 5y - 19 = 0$
25. $x^2 - 3x - 3y + 1 = 0$
26. $y^2 + 4y + x + 3 = 0$
27. $y^2 + 6y + \frac{1}{2}x + 7 = 0$
28. $x^2 + 2x - 3y + 19 = 0$
29. $x^2 + 2x + 2y + 3 = 0$
30. $y^2 - 6y + 2x + 17 = 0$

In Exercises 31–40 determine the equation of the parabola that has its vertex at the origin and that satisfies the given conditions.

*31. focus at $(1, 0)$

*32. focus at $(0, -3)$

*33. directrix $x = -\frac{3}{2}$

*34. directrix $y = \frac{5}{2}$

*35. axis is the x-axis, and parabola passes through the point $(2, 1)$

*36. axis is the y-axis, and parabola passes through the point $(4, -2)$

*37. axis is the x-axis, and $p = -\frac{5}{4}$

*38. axis is the y-axis, and $p = 2$

*39. focus at $(-1, 0)$ and directrix $x = 1$

*40. focus at $(0, -\frac{5}{2})$ and directrix $y = \frac{5}{2}$

12.5 The Ellipse and the Hyperbola

The Ellipse

The geometric definition of an ellipse is as follows:

> An **ellipse** is the set of all points the sum of whose distances from two fixed points is a constant.

The fixed points are called the **foci** of the ellipse. An ellipse may be constructed in the following way. Place a thumbtack at each of the foci F_1 and F_2, and attach one end of a string to each of the thumbtacks. Hold a pencil tight against the string, as shown in Figure 16, and move the pencil. The point P will describe an ellipse since the sum of the distances from P to the foci is always a constant, namely, the length of the string.

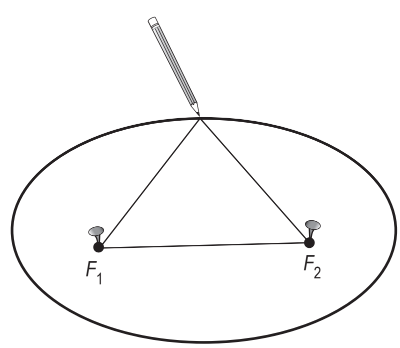


Figure 16 **Sketching an Ellipse**

The ellipse is in **standard position** if the foci are on either the x-axis or the y-axis and are equidistant from the origin. If the focus F_2 is at $(c, 0)$, then the other focus F_1 is at $(-c, 0)$, as in Figure 17. Let $P(x, y)$ be a point on the ellipse, and let the constant sum of the distances from P to the foci be denoted by $2a$. Then we have

$$\overline{PF_1} + \overline{PF_2} = 2a$$

Using the distance formula, we can obtain the equation of an ellipse in standard position as follows:

Standard Form of the Equation of an Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b \leq a$$

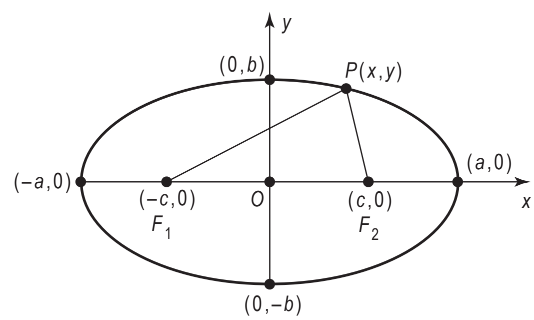


Figure 17 **Deriving the Equation of the Ellipse**

If we let $x = 0$ in the standard form, we find $y = \pm b$; if we let $y = 0$, we find $x = \pm a$. Thus, the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has intercepts $(\pm a, 0)$ and $(0, \pm b)$. See Figure 17.

EXAMPLE 1 USING THE STANDARD FORM FOR AN ELLIPSE

Find the intercepts and sketch the graph of the ellipse whose equation is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

SOLUTION

The intercepts are found by setting $x = 0$ and solving, then setting $y = 0$ and solving. Thus, the intercepts are $(\pm 4, 0)$ and $(0, \pm 3)$. The graph is then easily sketched (Figure 18).

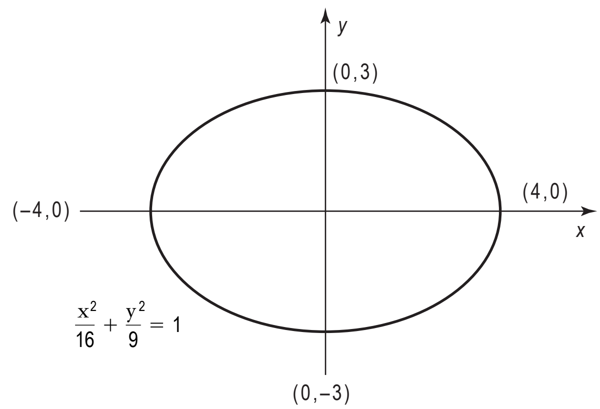


Figure 18 **Diagram for Example 1**

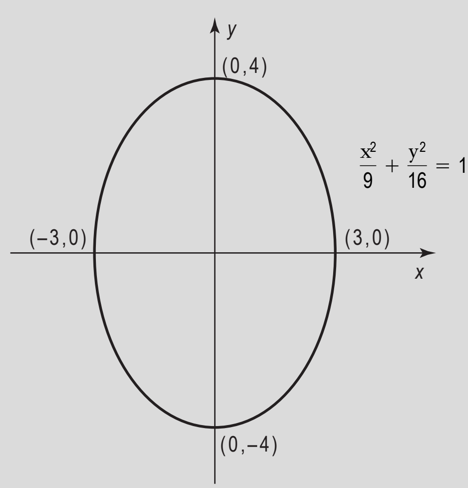
✔ Progress Check 1

Find the intercepts and sketch the graph of

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Answer

$(\pm 3, 0), (0, \pm 4)$



EXAMPLE 2 WRITING AN EQUATION IN STANDARD FORM

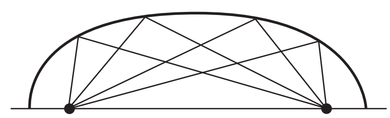
Write the equation of the ellipse in standard form and determine the intercepts.

(a) $4x^2 + 3y^2 = 12$ (b) $9x^2 + y^2 = 10$

Whispering Galleries

The domed roof in the accompanying figure has the shape of an ellipse that has been rotated about its major axis. It can be shown, using basic laws of physics, that a sound uttered at one focus will be reflected to the other focus, where it will be clearly heard. This property of such rooms is known as the "whispering gallery effect."

Famous whispering galleries include the dome of St. Paul's Cathedral, London; St. John Lateran, Rome; the Salle des Cariatides in the Louvre, Paris; and the original House of Representatives (now the National Statuary Hall in the United States Capitol), Washington, D.C.



SOLUTIONS

(a) Dividing by 12 to make the right-hand side equal to 1, we have

$$\frac{x^2}{3} + \frac{y^2}{4} = 1$$

The x-intercepts are $(\pm\sqrt{3}, 0)$; the y-intercepts are $(0, \pm 2)$.

(b) Dividing by 10, we have

$$\frac{9x^2}{10} + \frac{y^2}{10} = 1$$

But this is not standard form. However, if we write

$$\frac{9x^2}{10} \quad \text{as} \quad \frac{x^2}{\frac{10}{9}}$$

then

$$\frac{x^2}{\frac{10}{9}} + \frac{y^2}{10} = 1$$

is the standard form of an ellipse. The intercepts are

$$\left(\frac{\pm\sqrt{10}}{3}, 0\right) \quad \text{and} \quad (0, \pm\sqrt{10})$$

✔ Progress Check 2

Find the standard form and determine the intercepts of the ellipse.

(a) $2x^2 + 3y^2 = 6$ (b) $3x^2 + y^2 = 5$

Answers

(a) $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$ $(\pm\sqrt{3}, 0), (0, \pm\sqrt{2})$

(b) $\dfrac{x^2}{\frac{5}{3}} + \dfrac{y^2}{5} = 1$ $\left(\dfrac{\pm\sqrt{15}}{3}, 0\right), (0, \pm\sqrt{5})$

The Hyperbola

The hyperbola is the remaining conic section that we will consider in this chapter.

> A **hyperbola** is the set of all points the difference of whose distances from two fixed points is a positive constant.

The two fixed points are called the foci of the hyperbola, and the hyperbola is in **standard position** if the foci are on either the x-axis or the y-axis and are equidistant from the origin. If the foci lie on the x-axis and one focus F_2 is at $(c, 0)$, $c > 0$, then the other focus F_1 is at $(-c, 0)$. (See Figures 19a and 19b.)

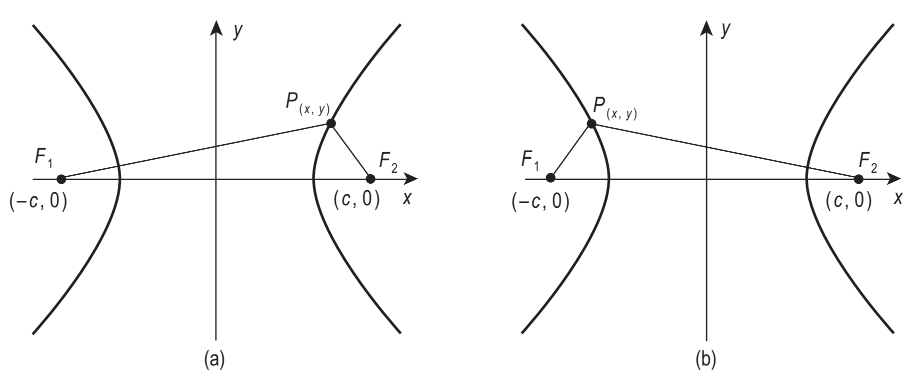


Figure 19 **Deriving the Equation of the Hyperbola**

Let P(x, y) be a point on the hyperbola, and let the constant difference of the distances from P to the foci be denoted by $2a$. If P is on the right branch, we have

$$\overline{PF_1} - \overline{PF_2} = 2a$$

whereas if P is on the left branch, we have

$$\overline{PF_2} - \overline{PF_1} = 2a$$

Both of these equations can be expressed by the single equation

$$|\overline{PF_1} - \overline{PF_2}| = 2a$$

Using the distance formula, we can obtain the equation of a hyperbola in standard position as follows:

Standard Form of the Equation of a Hyperbola (Foci on the *x*-axis)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1)$$

If the foci lie on the y-axis and one focus F_2 is at $(0, c)$, $c > 0$, then the other focus F_1 is at $(0, -c)$. In this case, we obtain the following equation of a hyperbola in standard position:

Standard Form of the Equation of a Hyperbola (Foci on the *y*-axis)

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad (2)$$

Letting $y = 0$, we see that the x-intercepts of the graph of Equation (1) are $\pm a$. Letting $x = 0$, we find there are no y-intercepts since the equation $y^2 = -b^2$ has no real roots. (See Figure 20.) Similarly, the graph of Equation (2) has y-intercepts $\pm a$ and no x-intercepts.

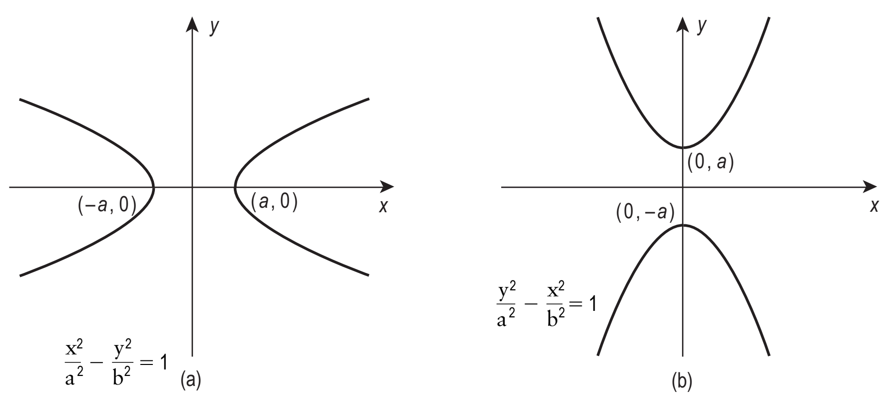


Figure 20 **Intercepts of the Hyperbola**

EXAMPLE 3 GRAPHING A HYPERBOLA

Find the intercepts and sketch the graph of the equation.

(a) $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ (b) $\dfrac{y^2}{4} - \dfrac{x^2}{3} = 1$

SOLUTIONS

(a) When $y = 0$, we have $x^2 = 9$, or $x = \pm 3$. The intercepts are $(3, 0)$ and $(-3, 0)$. With the assistance of a few plotted points, we can sketch the graph (Figure 21).

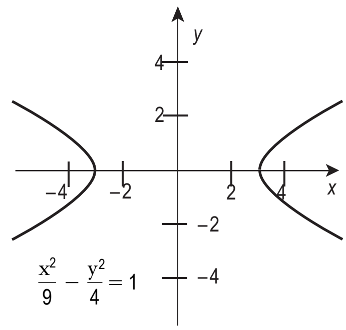


Figure 21 **Diagram for Example 3a**

(b) When $x = 0$, we have $y^2 = 4$, or $y = \pm 2$. The intercepts are (0, 2) and (0, −2). Plotting a few points, we can sketch the graph (Figure 22).

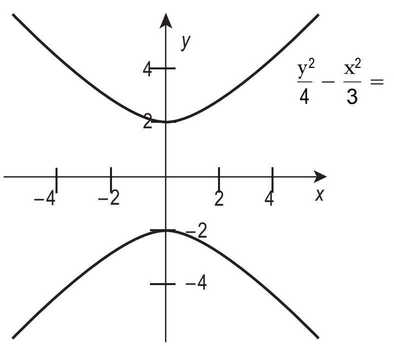


Figure 22 **Diagram for Example 3b**

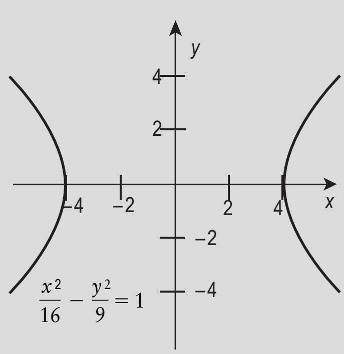
✔ Progress Check 3

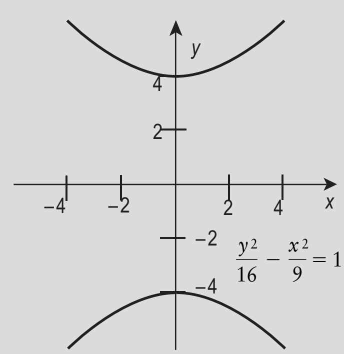
Find the intercepts and sketch the graph.

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (b) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

Answers

(a) intercepts are (4, 0) and (−4, 0) (b) intercepts are (0, 4) and (0, −4)





EXAMPLE 4 WRITING AN EQUATION IN STANDARD FORM

Write the equation of the hyperbola in standard form and determine the intercepts.

(a) $9y^2 - 4x^2 = 36$ (b) $8x^2 - 9y^2 = 18$

SOLUTIONS

(a) Dividing by 36 to produce a 1 on the right-hand side, we have

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

The y-intercepts are $(0, \pm 2)$. There are no x-intercepts.

(b) Dividing by 18, we have

$$\frac{4x^2}{9} - \frac{y^2}{2} = 1$$

Rewritten in standard form, the equation becomes

$$\frac{x^2}{\frac{9}{4}} - \frac{y^2}{2} = 1$$

The x-intercepts are

$$\left(\pm\frac{3}{2}, 0\right)$$

There are no y-intercepts.

✔ Progress Check 4

Write the equation of the hyperbola in standard form and determine the intercepts.

(a) $2x^2 - 5y^2 = 6$ (b) $4y^2 - x^2 = 5$

Answers

(a) $\dfrac{x^2}{3} - \dfrac{y^2}{\frac{6}{5}} = 1$ $(\pm\sqrt{3}, 0)$ (b) $\dfrac{y^2}{\frac{5}{4}} - \dfrac{x^2}{5} = 1$ $\left(0, \dfrac{\pm\sqrt{5}}{2}\right)$

Asymptotes

There is a way of sketching the graph of a hyperbola without the need for plotting points of the curve. Given the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in standard form, we plot the four points $(\pm a, \pm b)$, as in Figure 23, and draw the diagonals of the rectangle formed by the four points. The hyperbola opens from the intercepts $(\pm a, 0)$ and *approaches the lines formed by the diagonals of* *the rectangle*. We call these lines the asymptotes of the hyperbola. Since one asymptote passes through the points (0, 0) and (a, b), its equation is

$$y = \frac{b}{a}x$$

The equation of the other asymptote is found to be

$$y = -\frac{b}{a}x$$

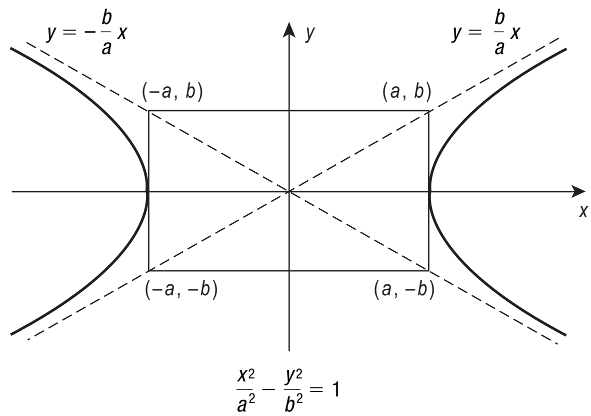


Figure 23 **The Asymptotes of a Hyperbola**

Of course, a similar argument can be made about the standard form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

In this case, the four points ($\pm b$, $\pm a$) determine the rectangle and the equations of the asymptotes are

$$y = \pm\frac{a}{b}x$$

To summarize:

Asymptotes of the Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	has asymptotes	$y = \pm\frac{b}{a}x$
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	has asymptotes	$y = \pm\frac{a}{b}x$

EXAMPLE 5 USING THE ASYMPTOTES TO GRAPH A HYPERBOLA

Using asymptotes, sketch the graph of the equation

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

SOLUTION

The points $(\pm 3, \pm 2)$ form the vertices of the rectangle. See Figure 24. Using the fact that $(0, \pm 2)$ are intercepts, we can sketch the graph opening from these points and approaching the asymptotes.

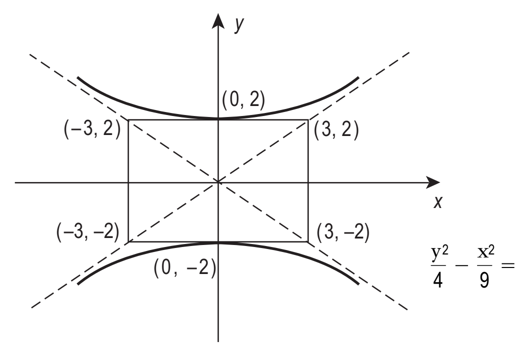
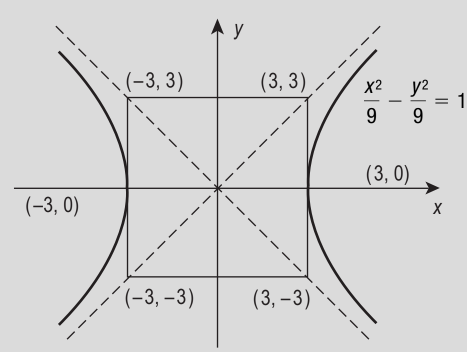


Figure 24 **Graph for Example 5**

✔ Progress Check 5

Using asymptotes, sketch the graph of the equation $\frac{x^2}{9} - \frac{y^2}{9} = 1$.

Answer



Exercise Set 12.5

In Exercises 1–8 find the intercepts and sketch the graph of the ellipse.

1. $\frac{x^2}{25} + \frac{y^2}{4} = 1$
2. $\frac{x^2}{4} + \frac{y^2}{16} = 1$
3. $\frac{x^2}{9} + \frac{y^2}{4} = 1$
4. $\frac{x^2}{12} + \frac{y^2}{18} = 1$
5. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
6. $\frac{x^2}{1} + \frac{y^2}{3} = 1$
7. $\frac{x^2}{20} + \frac{y^2}{10} = 1$
8. $\frac{x^2}{6} + \frac{y^2}{24} = 1$

In Exercises 9–18 write the equation of the ellipse in standard form and determine the intercepts.

9. $4x^2 + 9y^2 = 36$
10. $16x^2 + 9y^2 = 144$
11. $4x^2 + 16y^2 = 16$
12. $25x^2 + 4y^2 = 100$
13. $4x^2 + 16y^2 = 4$
14. $8x^2 + 4y^2 = 32$
15. $8x^2 + 6y^2 = 24$
16. $5x^2 + 6y^2 = 50$
17. $36x^2 + 8y^2 = 9$
18. $5x^2 + 4y^2 = 45$

In Exercises 19–26 find the intercepts and sketch the graph of the hyperbola.

19. $\frac{x^2}{25} - \frac{y^2}{16} = -1$
20. $\frac{y^2}{9} - \frac{x^2}{4} = 1$
21. $\frac{x^2}{36} - \frac{y^2}{9} = 1$
22. $\frac{y^2}{49} - \frac{x^2}{25} = 1$
23. $\frac{x^2}{6} - \frac{y^2}{8} = -1$
24. $\frac{y^2}{8} - \frac{x^2}{10} = -1$
25. $\frac{x^2}{12} - \frac{y^2}{2} = 1$
26. $\frac{y^2}{6} - \frac{x^2}{5} = 1$

In Exercises 27–36 write the equation of the hyperbola in standard form and determine the intercepts.

27. $16x^2 - y^2 = 64$
28. $4x^2 - 25y^2 = 100$
29. $4y^2 - 4x^2 = 1$
30. $2x^2 - 3y^2 = 6$
31. $4x^2 - 5y^2 = 20$
32. $25y^2 - 16x^2 = 400$
33. $4y^2 - 16x^2 = 64$
34. $35x^2 - 9y^2 = 45$
35. $8x^2 - 4y^2 = 32$
36. $4y^2 - 36x^2 = 9$

In Exercises 37–48, using asymptotes, sketch the graph of the given hyperbola.

37. $\frac{x^2}{16} - \frac{y^2}{4} = 1$
38. $\frac{y^2}{4} - \frac{x^2}{25} = 1$
39. $\frac{y^2}{4} - \frac{x^2}{4} = 1$
40. $\frac{x^2}{4} - \frac{y^2}{16} = 1$
41. $16x^2 - 4y^2 = 144$
42. $16y^2 - 25x^2 = 400$
43. $9y^2 - 9x^2 = 1$
44. $25x^2 - 9y^2 = 225$
45. $\frac{x^2}{25} - \frac{y^2}{36} = 1$
46. $y^2 - 4x^2 = 4$
47. $y^2 - x^2 = 1$
48. $\frac{x^2}{36} - \frac{y^2}{36} = 1$

In Exercises 49 and 50 find an equation of the ellipse satisfying the given conditions.

*49. Its intercepts are $(\pm 7, 0)$, and it passes through the point $\left(1, \frac{6\sqrt{3}}{7}\right)$.

*50. Its intercepts are $(0, \pm 1)$, and it passes through the point $\left(\frac{1}{4}, \frac{\sqrt{3}}{2}\right)$.

In Exercises 51–54 determine whether the foci of the given hyperbola lie on the x-axis or on the y-axis.

*51. $2x^2 - 3y^2 - 5 = 0$
*52. $3x^2 - 3y^2 + 4 = 0$
*53. $y^2 - 4x^2 - 20 = 0$
*54. $4y^2 - 9x^2 + 36 = 0$

In Exercises 55–58 find the equation of the hyperbola satisfying the given conditions.

*55. Its intercepts are $(0, \pm 3)$, and it has asymptote $y = x$.

*56. Its intercepts are $(\pm 2, 0)$, and it has asymptote $y = -2x$.

*57. Its intercepts are $(0, \pm 4)$, and it passes through the point $(5, 5)$.

*58. Its intercepts are $(\pm 2, 0)$, and it passes through the point $(3, 1)$.

59. Given the standard equation for an ellipse, define $c = \sqrt{a^2 - b^2}$. The ratio $\frac{c}{a}$ is called the eccentricity of the ellipse, and it measures how nearly circular the ellipse is. Determine the eccentricity of each ellipse in Exercises 1–8.

60. The planets in our solar system travel in elliptical orbits around the sun. The eccentricity of the Earth's orbit is about 0.017, and that of the orbit of Mars is 0.093. What does this tell you about the shapes of the two planets' orbits?

12.6 Identifying the Conic Sections

Each of the conic sections we have studied in this chapter has one or more axes of symmetry. We studied the circle and parabola when their axes of symmetry were the coordinate axes or lines parallel to them. Although our study of the ellipse and hyperbola was restricted to those that have the coordinate axes as their axes of symmetry, the same method of completing the square allows us to transform the general equation of the conic section

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

into standard form. This transformation is very helpful in sketching the graph of the conic section. It is easy, also, to identify the conic section from the general equation (see Table 2).

TABLE 2 The General Second-Degree Equation and the Conic Sections

$Ax^2 + Cy^2 + Dx + Ey + F = 0$	Conic Section	Remarks
$A = 0$ or $C = 0$ (A and C not both 0)	Parabola	Second degree in one variable, first degree in the other
$A = C \quad (\neq 0)$	Circle	Coefficients A and C are the same. *Caution:* Complete the square and check that $r > 0$.
$A \neq C$, $AC > 0$	Ellipse	A and C are unequal but have the same sign. *Caution:* Complete the square and check that the right-hand side is a positive constant.
$A \neq C$, $AC < 0$	Hyperbola	A and C have opposite signs.

EXAMPLE 1 IDENTIFYING THE CONIC SECTIONS

Identify the conic section.

(a) $3x^2 + 3y^2 - 2y = 4$
(b) $3x^2 - 9y^2 + 2x - 4y = 7$
(c) $2x^2 + 5y^2 - 7x + 3y - 4 = 0$
(d) $3y^2 - 4x + 17y = -10$

SOLUTIONS

(a) Since the coefficients of x^2 and y^2 are the same, the graph will be a circle if the standard form yields $r > 0$. Completing the square, we have

$$3x^2 + 3\left(y - \frac{1}{3}\right)^2 = \frac{13}{3}$$

which is the equation of a circle.

(b) Since the coefficients of x^2 and y^2 are of opposite sign, the graph is a hyperbola.

(c) The coefficients of x^2 and y^2 are unequal but of like sign, so the graph is an ellipse. (Verify that the right-hand side is positive.)

(d) The graph is a parabola since the equation is of the second degree in y and of first degree in x.

✔ Progress Check 1

Identify the conic section.

(a) $\frac{x^2}{5} - 3y^2 - 2x + 2y - 4 = 0$ (b) $x^2 - 2y - 3x = 2$

(c) $x^2 + y^2 - 4x - 6y = -11$ (d) $4x^2 + 3y^2 + 6x - 10 = 0$

Answers

(a) hyperbola (b) parabola (c) circle (d) ellipse

A summary of the characteristics of the conic sections is given in Table 3.

TABLE 3 Standard Forms of the Conic Sections

Curves and Standard Equation	Characteristics	Example
Circle $(x - h)^2 + (y - k)^2 = r^2$	Center: (h, k) Radius: r	$(x - 2)^2 + (y + 4)^2 = 25$ Center: $(2, -4)$ Radius: 5
Parabola $(x - h)^2 = 4p(y - k)$	Vertex: (h, k) Axis: $x = h$ $p > 0$: Opens up $p < 0$: Opens down	$(x + 1)^2 = 2(y - 3)$ Vertex: $(-1, 3)$ Axis: $x = -1$ Opens up
or		
$(y - k)^2 = 4p(x - h)$	Vertex: (h, k) Axis: $y = k$ $p > 0$: Opens to the right $p < 0$: Opens to the left	$(y + 4)^2 = -3(x + 5)$ Vertex: $(-5, -4)$ Axis: $y = -4$ Opens to the left
Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Intercepts: $(\pm a, 0)$, $(0, \pm b)$	$\frac{x^2}{4} + \frac{y^2}{6} = 1$ Intercepts: $(\pm 2, 0)$, $(0, \pm \sqrt{6})$

continues

TABLE 3 *continued*

Curves and Standard Equation	Characteristics	Example
Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Intercepts: $(\pm a, 0)$ Asymptotes: $y = \pm\frac{b}{a}x$ Opens to left and right	$\frac{x^2}{4} - \frac{y^2}{9} = 1$ Intercepts: $(\pm 2, 0)$ Asymptotes: $y = \pm\frac{3}{2}x$ Opens to left and right
or		
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	Intercepts: $(0, \pm a)$ Asymptotes: $y = \pm\frac{a}{b}x$ Opens up and down	$\frac{y^2}{9} - \frac{x^2}{4} = 1$ Intercepts: $(0, \pm 3)$ Asymptotes: $y = \pm\frac{3}{2}x$ Opens up and down

Exercise Set 12.6

In Exercises 1–30 identify the conic section.

1. $2x^2 + y - x + 3 = 0$
2. $4y^2 - x^2 + 2x - 3y + 5 = 0$
3. $4x^2 + 4y^2 - 2x + 3y - 4 = 0$
4. $3x^2 + 6y^2 - 2x + 8 = 0$
5. $36x^2 - 4y^2 + x - y + 2 = 0$
6. $x^2 + y^2 - 6x + 4y + 13 = 0$
7. $16x^2 + 4y^2 - 2y + 3 = 0$
8. $2y^2 - 3x + y + 4 = 0$
9. $x^2 + y^2 - 4x - 2y + 8 = 0$
10. $x^2 + y^2 - 2x - 2y + 6 = 0$
11. $4x^2 + 9y^2 - x + 2 = 0$
12. $3x^2 + 3y^2 - 3x + y = 0$
13. $4x^2 - 9y^2 + 2x + y + 3 = 0$
14. $x^2 + y^2 + 6x - 2y + 10 = 0$
15. $x^2 + y^2 - 4x + 4 = 0$
16. $2x^2 + 3x - 5y^2 + 4y - 6 = 0$
17. $4x^2 + y^2 - 2x + y + 4 = 0$
18. $x^2 - \frac{1}{2}y^2 + 2x - y + 3 = 0$
19. $4x^2 + 4y^2 - x + 2y - 1 = 0$
20. $x^2 + y^2 + 6x - 6y + 18 = 0$
21. $4x^2 + y + 2x - 3 = 0$
22. $y^2 - \frac{1}{4}x^2 + 2x + 6 = 0$
23. $x^2 + y^2 + 4x - 2y + 7 = 0$
24. $y^2 + 2y - \frac{1}{2}x + 3 = 0$
25. $x^2 + y^2 - 2x - 10y - 26 = 0$
26. $3x^2 + 2y^2 - y + 2 = 0$
27. $\frac{1}{2}x^2 + y - x - 3 = 0$
28. $3x^2 - 2y^2 + 2x - 5y + 5 = 0$
29. $2x^2 + y^2 - 3x + 2y - 5 = 0$
30. $x^2 - 2x - y + 1 = 0$

■ ■ ■

Chapter Summary

Terms and Symbols

Key Ideas for Review

Topic	Page	Key Idea
Distance and midpoint formulas	443	If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points, then the distance d between the points is given by the formula $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ and the midpoint Q has coordinates $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
Analytic geometry: proving theorems	446	Theorems from plane geometry can be proven using the methods of analytic geometry. In general, place the given geometric figure in a convenient position relative to the origin and axes. The distance formula, the midpoint formula, and the computation of slope are the tools to apply in proving such theorems.
Symmetry	448	Symmetry about a line or a point means that the curve is its own reflection about that line or that point. The graph of an equation in x and y will be symmetric with respect to the (a) x-axis if an equivalent equation results when y is replaced by $-y$; (b) y-axis if an equivalent equation results when x is replaced by $-x$; (c) origin if an equivalent equation results when both x and y are replaced by $-x$ and $-y$, respectively
Conic sections	452	The conic sections are the circle, the parabola, the ellipse, and the hyperbola. In some special cases, these reduce to a point, a line, two lines, or no graph. The conic sections represent the possible intersections of a cone and a plane.
geometric definition	452	Each conic section has a geometric definition as a set of points satisfying certain given conditions.

continues

Topic	Page	Key Idea
Graph of a second-degree equation		
standard form	478	A second-degree equation in x and y can be written in standard form by completing the square in the variables. It is much easier to sketch the graph when the equation is in standard form than it is when the equation is in general form.
general form	477	It is often possible to distinguish the various conic sections even when the equation is given in the general form.

Common Errors

1. When completing the square, be careful to balance both sides of the equation.

2. The first step in writing the equation

$$4x^2 + 25y^2 = 9$$

in standard form is to divide both sides of the equation by 9. The result,

$$\frac{4x^2}{9} + \frac{25y^2}{9} = 1$$

is *not* in standard form. You must rewrite this as

$$\frac{x^2}{\frac{9}{4}} + \frac{y^2}{\frac{9}{25}} = 1$$

to obtain standard form and to determine the intercepts

$$\left(\pm\frac{3}{2}, 0\right), \left(0, \pm\frac{3}{5}\right).$$

3. The graph of the equation $3y^2 - 4x - 6 = 0$ is *not* a hyperbola. If only one variable appears to the second degree, the equation is that of a parabola.

4. The safest way to find the intercepts is to let one variable equal 0 and solve for the other variable. If you attempt to memorize the various forms, you might conclude that the intercepts of

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

are $(\pm 4, 0)$. However, when $x = 0$, we see that $y^2 = 16$ or $y = \pm 4$ and the intercepts are $(0, \pm 4)$. To find the intercepts of

$$\frac{y^2}{16} - \frac{x^2}{9} = -1$$

don't conclude that the intercepts are $(0, \pm 4)$. When $x = 0$, $y^2 = -16$ has no solution. When $y = 0$, $x^2 = 9$ leads to the intercepts $(\pm 3, 0)$.

5. When analyzing the type of conic section from the general form of the second-degree equation, remember that the circle and ellipse have degenerate cases in which the graph turns out to be a point, a line, or a pair of lines. The equation

$$2x^2 + y^2 - 8x + 6y + 21 = 0$$

is equivalent to

$$2(x - 2)^2 + (y + 3)^2 = -4$$

which is impossible. There are no points we can graph that will satisfy this equation.

Review Exercises

Solutions to exercises whose numbers are in bold are in the Solutions section in the back of the book.

12.1 In Exercises 1–3 find the distance between the given pair of points.

1. $(-4, -6), (2, -1)$ **2.** $(3, 4), (3, -2)$

3. $(4, 5), (1, 3)$

In Exercises 4–6 find the midpoint of the line segment whose endpoints are given.

4. $(-5, 4), (3, -6)$ 5. $(-2, 0), (-3, 5)$

6. $(2, -7), (-3, -2)$

7. Find the coordinates of the point P_2 if $(2, 2)$ are the coordinates of the midpoint of the line segment joining $P_1(-6, -3)$ and P_2.

8. Use the distance formula to show that $P_1(-1, 2)$, $P_2(4, 3)$, $P_3(1, -1)$, and $P_4(-4, -2)$ are the coordinates of a parallelogram.

9. Show that the points $A(-8, 4)$, $B(5, 3)$, and $C(2, -2)$ are the vertices of a right triangle.

10. Find an equation of the perpendicular bisector of the line segment joining the points $A(-4, -3)$ and $B(1, 3)$. (The perpendicular bisector passes through the midpoint of AB and is perpendicular to AB.)

12.2 In Exercises 11 and 12 analyze the given equation for symmetry with respect to the x-axis, y-axis, and origin.

11. $y^2 = 1 - x^3$ 12. $y^2 = \dfrac{x^2}{x^2 - 5}$

12.3 13. Write an equation of the circle with center at $(-5, 2)$ and a radius of 4.

14. Write an equation of the circle with center at $(-3, -3)$ and radius 2.

In Exercises 15–20 determine the center and radius of the circle with the given equation.

15. $(x - 2)^2 + (y + 3)^2 = 9$

16. $\left(x + \dfrac{1}{2}\right)^2 + (y - 4)^2 = \dfrac{1}{9}$

17. $x^2 + y^2 + 4x - 6y = -10$

18. $2x^2 + 2y^2 - 4x + 4y = -3$

19. $x^2 + y^2 - 6y + 3 = 0$

20. $x^2 + y^2 - 2x - 2y = 8$

12.4 In Exercises 21 and 22 determine the vertex and axis of the given parabola. Sketch the graph.

21. $(y + 5)^2 = 4\left(x - \dfrac{3}{2}\right)$ 22. $(x - 1)^2 = 2 - y$

In Exercises 23–28 determine the vertex, axis, and direction of the given parabola.

23. $y^2 + 3x + 9 = 0$

24. $y^2 + 4y + x + 2 = 0$

25. $2x^2 - 12x - y + 16 = 0$

26. $x^2 + 4x + 2y + 5 = 0$

27. $y^2 - 2y - 4x + 1 = 0$

28. $x^2 + 6x + 4y + 9 = 0$

12.5 In Exercises 29–34 write the given equation in standard form and determine the intercepts of its graph.

29. $9x^2 - 4y^2 = 36$ 30. $9x^2 + y^2 = 9$

31. $5x^2 + 7y^2 = 35$ 32. $9x^2 - 16y^2 = 144$

33. $3x^2 + 4y^2 = 9$ 34. $3y^2 - 5x^2 = 20$

In Exercises 35 and 36 use the intercepts and asymptotes of the hyperbola to sketch the graph.

35. $4x^2 - 4y^2 = 1$ 36. $9y^2 - 4x^2 = 36$

12.6 In Exercises 37–40 identify the conic section whose equation is given.

37. $2y^2 + 6y - 3x + 2 = 0$

38. $6x^2 - 7y^2 - 5x + 6y = 0$

39. $2x^2 + y^2 + 12x - 2y + 17 = 0$

40. $9x^2 + 4y^2 = -36$

Progress Test 12A

1. Find the distance between the points $P_1(-3, 4)$ and $P_2(4, -3)$.
2. Find the midpoint of the line segment whose endpoints are $P_1(\frac{1}{2}, -1)$ and $P_2(2, \frac{3}{4})$.
3. Given the points $A(1, -2)$, $B(5, -1)$, $C(2, 7)$, and $D(6, 8)$, show that AC is equal and parallel to BD.
4. Show that $A(-1, 7)$, $B(-3, 2)$, and $C(4, 5)$ are the coordinates of the vertices of an isosceles triangle.
5. Without sketching, determine symmetry with respect to the x-axis, y-axis, and origin:
$$3x^2 - 2x - 4y^2 = 6$$
6. Without sketching, determine symmetry with respect to the x-axis, y-axis, and origin:
$$y^2 = \frac{2x}{x^2 - 1}$$
7. Find the center and radius of the circle whose equation is $x^2 - 8x + y^2 + 6y + 15 = 0$. Sketch.
8. Find the vertex and axis of symmetry of the parabola whose equation is $4y^2 - 4y + 12x = -13$. Sketch.
9. Find the intercepts and asymptotes of the hyperbola whose equation is
$$4y^2 - \frac{x^2}{9} = 1.$$
Sketch.
10. Find the equation of the circle having center at $(\frac{2}{3}, -3)$ and radius $\sqrt{3}$.
11. Find the intercepts and vertex and sketch the graph of the parabola whose equation is $y = x^2 - x - 6$.
12. Find the intercepts and sketch the graph of the equation
$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$
13. Identify the conic section whose equation is
$$3x^2 + 2x - 7y^2 + 3y - 14 = 0$$
14. Identify the conic section whose equation is
$$y^2 - 3y - 5x = 20$$
15. Identify the conic section whose equation is
$$x^2 - 6x + y^2 = 0$$

Progress Test 12B

1. Find the distance between the points $P_1(6, -7)$ and $P_2(2, -5)$.
2. Find the midpoint of the line segment whose endpoints are $P_1(-\frac{3}{2}, -\frac{1}{2})$ and $P_2(-2, 1)$.
3. The point (3, 2) is the midpoint of a line segment having the point (4, −1) as an endpoint. Find the other endpoint.
4. Show that the points $A(-8, 4)$, $B(5, 3)$, and $C(2, -2)$ are the vertices of a right triangle.
5. Without sketching, determine symmetry with respect to the x-axis, y-axis, and origin:
$$y = 3x^3 - 8x$$
6. Without sketching, determine symmetry with respect to the x-axis, y-axis, and origin:
$$y = \frac{1}{4 - x^2}$$
7. Find the center and radius of the circle whose equation is $x^2 + x + y^2 - 6y = -9$. Sketch.
8. Find the vertex and axis of symmetry of the parabola whose equation is $16x^2 - 8x + 32y + 65 = 0$. Sketch.
9. Find the intercepts and asymptotes of the hyperbola whose equation is $3x^2 - 8y^2 = 2$. Sketch.
10. Find the equation of the circle having center at $(-1, -\frac{1}{2})$ and radius $\sqrt{5}$.
11. Find the intercepts and vertex and sketch the graph of the parabola whose equation is $y = -x^2 + 16x - 14$.
12. Find the intercepts and sketch the graph of the equation
$$4x^2 + y^2 = 9$$

13. Identify the conic section whose equation is

$$x^2 - 4x + y^2 - 6y + 12 = 0$$

14. Identify the conic section whose equation is

$$x^2 + 2x - 5y + 4 = 0$$

15. Identify the conic section whose equation is

$$x^2 + 2y^2 - 4x + 3 = 0$$

Chapter 12 Project

The planet that is usually farthest from the sun is Pluto. The orbit of Pluto, like that of all the planets in our Solar System, is an ellipse. The ellipse that Pluto traces, however, is much less nearly circular than that of any other planet. The eccentricity of the orbit of Pluto is about 0.25, which is almost fifteen times as great as that of Earth.

Review Exercises 59 and 60 in Section 12.5.

For this project, we will imagine ourselves to be NASA scientists, planning to launch a probe to study Pluto. We must try to determine an equation for the orbit of Pluto, given the following facts: The eccentricity of the ellipse is 0.25, and the value of a is about 3,670 million miles. Recall that $e = \frac{c}{a}$.

From this information, set up and solve an equation for c. Now use the fact that $c = \sqrt{a^2 - b^2}$ to solve a radical equation for b. Then write out an equation for the ellipse that is the orbit of Pluto.

Do some more research into the mathematics of space and space travel. How accurate is your equation?

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